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ROYAL AIRCRAFT ESTABLISHMENT
(FARNBOROUGH)

TECHNICAL NOTE No: G.W. 567

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A METHOD OF CALCULATING THE
VACUUM SPECIFIC IMPULSE OF BALLISTIC
MISSILES FROM LONGITUDINAL
ACCELERATION RECORDS

by

D. A. BERRY

PICATINNY ARSENAL
TECHNICAL INFORMATION SECTION

MARCH, 1961

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March, 1961

ROYAL AIRCRAFT ESTABLISHMENT, & B

(FARNBOROUGH)

A METHOD OF CALCULATING THE VACUUM SPECIFIC IMPULSE OF BALLISTIC
MISSILES FROM LONGITUDINAL ACCELERATION RECORDS

by

D.A. Berry

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SUMMARY

A method of analysis is presented by means of which certain missile parameters, in particular the vacuum specific impulse, may be calculated using only the telemetry record of longitudinal acceleration. The accuracy of the method is discussed on a statistical basis and graphs are provided which may be used to predict the accuracy of calculation for a missile of known nominal performance.

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1 INTRODUCTION

During the course of the flight trials of a ballistic missile, it is necessary to measure the performance of the rocket engine. The specification limits for most ballistic missiles are so narrow as to demand a high order of accuracy of measurement of engine performance. One of the most important parameters is the specific impulse and the usefulness of the method presented in this Note lies in the fact that it allows the specific impulse to be derived directly. A considerable part of the Note is concerned with the accuracy of the method and graphs are given which may be used to predict the accuracy of measurement for a missile of known nominal performance.

2 APPARENT ACCELERATION IN VACUO

For missiles of range longer than a hundred miles or so the major part of the boost phase takes place outside the earth's atmosphere.

In this note, it will be assumed that the missile is in vacuum when the aerodynamic forces are negligibly small compared with the thrust of the motor so that neglecting these forces will produce an acceptably small error.

Outside the atmosphere only two forces act on the missile, one being the motor thrust, the other being that of gravitational attraction. The missile's true acceleration is due to both these forces acting together but to understand its behaviour more fully it is useful to resolve the 'true' acceleration into (i) an 'apparent' acceleration, due to motor thrust alone, and (ii) a 'gravitational' acceleration, due to the gravity field alone. In general these will act in different directions. If an accelerometer is mounted in the missile, then, during its flight in vacuo, it will measure only the apparent acceleration. This is because accelerometers measure acceleration by sensing the force which has to be exerted on a known mass in order that the mass moves with the same acceleration as the missile. However, gravitational attraction exerts forces on the mass in the accelerometer and on the missile mass so that the accelerations due to this cause are the same. Because there is no relative acceleration between the missile and the accelerometer arising from the earth's gravitational field, no force can arise in the accelerometer mounting system, hence a missile-borne accelerometer cannot measure that component of the missile's true acceleration which is due to gravity. Normally accelerometers have only one degree of freedom, i.e. they measure the component of the apparent acceleration in one direction, usually fixed in relation to the missile body axes. However, if the missile employs inertia navigation, then the accelerometers will be mounted on a table which will maintain a fixed direction in space independent of missile orientation.

3 EQUATION OF MOTION ALONG THE FLIGHT PATH

If f is the apparent acceleration along the flight path, then the equation of motion is at any instant

$$f = \frac{Tg}{m} \quad (1)$$

where T is the component of thrust along the flight path (lb wt)

m is the mass of the missile in lb

f is apparent acceleration in ft/sec^2

g is 32.2 ft/sec^2 .

It will be assumed that the thrust and the propellant consumption, \dot{m} , of the engine are constant. If at time $t = 0$, the mass of the missile is m_0 , then the mass of the missile at time t will be less due to the consumption of the propellants and can be written

$$m = m_0 - \dot{m} t$$

therefore

$$f = \frac{Tg}{m_0 - \dot{m} t}$$

By virtue of the above assumption, the specific impulse, I , which is defined by the ratio of thrust to mass flow, will also be constant. Hence

$$\begin{aligned} f &= \frac{Ig}{m_0/\dot{m} - t} \\ &= \frac{Ig}{b - t} \end{aligned} \quad (2)$$

where

$$b = m_0/\dot{m}$$

We assume that the apparent acceleration f measured aboard the missile is telemetered back to the ground. We also assume that the apparent acceleration is sampled along with other measurements so that the output from the ground receiver is a series of readings made at equal intervals of time. During the process of transmission from the missile to the ground receivers and of conversion to a suitable form, the data will suffer a certain amount of degradation. Furthermore, the accelerometer will respond to a certain extent to missile vibration. Consequently the data which is available is subject to error. However, we shall assume that, at least for the range of data over which the analysis is to be conducted, the output from the ground receivers is a linear function of the apparent acceleration. Hence

$$f = k(y - a)$$

where k is the constant of proportionality

a is the zero or bias error

and y is the ground receiver output.

Substituting for f in equation (2) we get

$$y = a + \frac{o}{b - t} \quad (3)$$

where

$$o = Ig/k \quad (4)$$

The method given below proposes that the coefficients ' a ', b and o may be determined by fitting a curve of the form of equation (3) to the telemetry readings, y , and the corresponding values of time, t . Normally ' a ' is measured during the pre-flight calibration of the accelerometer and

telemetry system. However, any bias errors arising in the telemetry system during flight will alter the value of 'a'. It may be desirable, therefore, to determine 'a' by the process of curve fitting. Thus, both the method of determining all three coefficients, 'a', b and c, and the method of determining b and c, given the value of 'a', to some estimated accuracy, are given below. The former case is referred to as the case of 'a' unknown and the latter as that of 'a' known.

The value of c determined by the proposed method may be used to determine the specific impulse, I, by substitution in the formula

$$I = \frac{ck}{g}$$

where the scale factor, k, will be determined to some estimated accuracy by the pre-flight calibration.

The coefficient, b, may be used to determine the mass ratio, λ , which is defined as the ratio of the all-up weight at launch ($t = 0$) to the weight at some time t after launch. Thus

$$\begin{aligned}\lambda &= \frac{m_0}{m_0 - \dot{m}t} \\ &= \frac{b}{b - t} \quad .\end{aligned}$$

The mass at time t may be evaluated if the mass at launch is known. Thus

$$\begin{aligned}m &= m_0 - \dot{m}t \\ &= m_0(1 - t/b) \quad .\end{aligned}$$

For a given accuracy of measurement of b and m_0 , the accuracy of determination of m will reduce linearly with time, so that unless very accurate determination of both m_0 and b are possible, the accuracy of determination of m toward the end of the boost phase is unlikely to be satisfactory.

Alternatively, the mass at time t may be determined in conjunction with an independent measurement of mass flow, \dot{m} , e.g. by flowmeter. Thus

$$\begin{aligned}m &= m_0 - \dot{m}t \\ &= \dot{m}(b - t) \quad .\end{aligned}$$

The accuracy of determining the mass by this method will not be dependent on time so that although the accuracy of measurement of \dot{m} in flight may not be as good as the accuracy of measurement of m_0 just before launch, the latter method is likely to be more satisfactory than the former in the later stages of boosted flight.

An independent measurement of the mass flow, \dot{m} , is also required to determine the thrust in conjunction with the measured value of the specific impulse, using the formula

$$T = I \dot{m} .$$

This method of deriving the thrust serves as an independent and overall check on the engine performance because it requires no prior knowledge of the engine performance characteristics.

4 THE METHOD OF DETERMINING THE COEFFICIENTS

The first case to be dealt with is the general case where 'a' is unknown. The more particular case where 'a' is known may then be obtained along similar lines (see section 7).

It will be assumed that although y will be subject to errors, measurement of time will be sufficiently accurate, and as a general rule this is so. In general a large number of observations of y and t will be made so that the problem is to determine the values of the coefficients a, b and c which best fit the observations. This will be done by the method of least squares¹. Because the function that must be fitted to the data is non-linear in t, it is necessary to use a process of differential adjustment in order to determine the coefficients. The method has therefore been treated in detail.

At a time t_1 , let the fitted value of y_1 be Y_1 . Thus when a, b and c have their most probable values, Y_1 is given by

$$Y_1 = a + \frac{c}{b - t_1} .$$

The residual, v_1 , at time, t_1 , is defined as the difference between the observed value y_1 and the fitted value Y_1 , i.e.

$$v_1 = y_1 - Y_1$$

therefore

$$v_1 = y_1 - a - \frac{c}{b - t_1} . \quad (4)$$

The method of least squares shows that if the true errors (i.e. the differences between the true and the observed values of y) are independent and normally distributed, the most probable values of a, b and c are those which make the sum of the squares of the residuals a minimum, i.e.

$$\left. \begin{aligned} \frac{\partial}{\partial a} \sum_{1}^n v_1^2 &= 0 \\ \frac{\partial}{\partial b} \sum_{1}^n v_1^2 &= 0 \\ \frac{\partial}{\partial c} \sum_{1}^n v_1^2 &= 0 \end{aligned} \right\} \quad (5)$$

These three equations, which are called the normal (or normalised) equations, are sufficient to determine a , b and c . Thus after squaring (4) and summing over all values of i and then partially differentiating with respect to a , b and c in turn we get

$$\left. \begin{aligned} \sum_{i=1}^n \left(y_i - a - \frac{c}{b - t_i} \right) &= 0 \\ \sum_{i=1}^n \left(y_i - a - \frac{c}{b - t_i} \right) \frac{1}{(b - t_i)^2} &= 0 \\ \sum_{i=1}^n \left(y_i - a - \frac{c}{b - t_i} \right) \frac{-1}{b - t_i} &= 0 \end{aligned} \right\} \quad (6)$$

The values of the coefficients which satisfy the above equations are the most probable values. However, it will be seen that the equations are non-linear in b and can only be solved by a process of successive approximation. Because it is desirable to calculate the variances of the coefficients, expressions for which have only been derived for linear normal equations, it is necessary to linearise equation (4) by the process of differential adjustment rather than to solve equations (6) directly.

The process of differential adjustment assumes that good estimates of b and c , say b_0 and c_0 respectively, can be obtained. Next the most probable values of adjustments β and γ , which must be made to b_0 and c_0 can be found, i.e. the values of a , β and γ can be found which make the sum of the squares of the residuals a minimum. It is supposed that the original estimates b_0 and c_0 are sufficiently good that β and γ can be regarded as small quantities.

In equation (4) putting $b = b_0 - \beta$ and $c = c_0 + \gamma$ we get

$$v_i = y_i - a - \frac{c_0}{(b_0 - t_i)} \left(1 + \frac{\gamma}{c_0} \right) \left(1 - \frac{\beta}{b_0 - t_i} \right)^{-1}.$$

If $\frac{\gamma}{c_0}$ and $\frac{\beta}{b_0 - t_i}$ are small compared with 1, the above equation becomes approximately

$$v_i = \left[y_i - \frac{c_0}{b_0 - t_i} \right] - a - \frac{\gamma}{(b_0 - t_i)} - \frac{c_0 \beta}{(b_0 - t_i)^2}.$$

Putting $b_0 - t_i = x_i$, we obtain

$$v_i = \left(y_i - \frac{c_0}{x_i} \right) - a - \frac{1}{x_i} \gamma - \frac{c_0}{x_i^2} \beta. \quad (7)$$

Squaring equation (7), summing for all available values of i and partially differentiating with respect to a , β and γ in turn we get

$$a \sum_1^n 1 + \beta \sum_1^n \frac{c_o}{x_1^2} + \gamma \sum_1^n \frac{1}{x_1} - \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) = 0 \quad (8)$$

$$a \sum_1^n \frac{c_o}{x_1^2} + \beta \sum_1^n \frac{c_o^2}{x_1^4} + \gamma \sum_1^n \frac{c_o}{x_1^3} - \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{c_o}{x_1^2} = 0 \quad (9)$$

$$a \sum_1^n \frac{1}{x_1} + \beta \sum_1^n \frac{c_o}{x_1^3} + \gamma \sum_1^n \frac{1}{x_1^2} - \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{1}{x_1} = 0 \quad (10)$$

The solution of these equations in determinantal form is

$$Da = A_{11} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) + A_{21} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{c_o}{x_1^2} + A_{31} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{1}{x_1} \quad (11)$$

$$D\beta = A_{12} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) + A_{22} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{c_o}{x_1^2} + A_{32} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{1}{x_1} \quad (12)$$

$$D\gamma = A_{13} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) + A_{23} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{c_o}{x_1^2} + A_{33} \sum_1^n \left(y_1 - \frac{c_o}{x_1} \right) \frac{1}{x_1} \quad (13)$$

where

$$D = \begin{vmatrix} \sum_1^n 1 & \sum_1^n \frac{c_o}{x_1^2} & \sum_1^n \frac{1}{x_1} \\ \sum_1^n \frac{c_o}{x_1^2} & \sum_1^n \frac{c_o^2}{x_1^4} & \sum_1^n \frac{c_o}{x_1^3} \\ \sum_1^n \frac{1}{x_1} & \sum_1^n \frac{c_o}{x_1^3} & \sum_1^n \frac{1}{x_1^2} \end{vmatrix} \quad (14)$$

and A_{pq} is the co-factor of the element in the p^{th} row and q^{th} column of the determinant D .

Having determined the values of a , β and γ , β and γ can be used to adjust b_o and c_o and the adjusted values used as a new estimate of b and c . Further adjustments can then be computed by the above method until b and c have been computed to the desired accuracy.

It is not proposed to consider, analytically, the convergence of the method. However on all the occasions when the method has been used, the convergence has been found to be satisfactory.

5 ESTIMATES OF THE VARIANCE OF THE COEFFICIENTS

Formulae for the variance of the coefficients have been derived¹ on the assumption that the errors are independent and normally distributed and that the distribution is independent of time. In the notation of the previous section these formulae are:-

$$s_a^2 = s^2 \frac{A_{11}}{D} \quad (15)$$

$$s_b^2 = s^2 \frac{A_{22}}{D} \quad (16)$$

$$s_o^2 = s^2 \frac{A_{33}}{D} \quad (17)$$

$$\text{where } s^2 = \frac{1}{m-3} \frac{E}{D} = \frac{\sum_{i=1}^n v_i^2}{m-3} \quad (18)$$

where s_a^2 = estimate of the variance of a
 s_b^2 = estimate of the variance of b
 s_o^2 = estimate of the variance of o
m = number of observations

and

$$E = \begin{vmatrix} \sum 1 & \sum \frac{o_o}{x_i} & \sum \frac{1}{x_i} & \sum \left(y_i - \frac{o_o}{x_i}\right) \\ \sum \frac{o_o}{x_i} & \sum \frac{o_o^2}{x_i^2} & \sum \frac{o_o}{x_i^2} & \sum \left(y_i - \frac{o_o}{x_i}\right) \frac{o_o}{x_i} \\ \sum \frac{1}{x_i} & \sum \frac{o_o}{x_i^2} & \sum \frac{1}{x_i^2} & \sum \left(y_i - \frac{o_o}{x_i}\right) \frac{1}{x_i} \\ \sum \left(y_i - \frac{o_o}{x_i}\right) & \sum \frac{o_o}{x_i^2} \left(y_i - \frac{o_o}{x_i}\right) & \sum \frac{1}{x_i} \left(y_i - \frac{o_o}{x_i}\right) & \sum \left(y_i - \frac{o_o}{x_i}\right)^2 \end{vmatrix} \quad (19)$$

It should be noted that the above estimates of the variance relate strictly to β or γ rather than to b or c . However as $b = b_0 - \beta$ and b_0 is a constant, then the variance of b is the same as the variance of β . Similarly the variance of c is equal to the variance of γ .

6 PREDICTION OF THE VARIANCE OF THE COEFFICIENTS

It will be seen from equation (14) that the determinant D and consequently the co-factors depend only on c_0 , b_0 and t_1 and is independent of y .

It is therefore possible to express the variance of the coefficients in terms of these variables after making certain simplifying assumptions. It is then possible to predict the variance of the coefficients approximately for a given missile and so obtain prior knowledge of the precision of the analysis.

In the determinant D there are only 5 terms, i.e. $\sum_{i=1}^n 1$, $\sum_{i=1}^n \frac{1}{x_i}$, $\sum_{i=1}^n \frac{1}{x_i^2}$, $\sum_{i=1}^n \frac{1}{x_i^3}$ and $\sum_{i=1}^n \frac{1}{x_i^4}$. It should be noted that c_0 is a common

factor in the second column and so may be brought outside the determinant. Also, by interchanging the last two columns and then the last two rows, the determinant becomes symmetrical about the leading diagonal.

We now assume that the observations are made at uniform intervals of time δt and that the interval is small. Consequently

$$\begin{aligned} \sum_{i=1}^n \frac{1}{x_i^p} &= \frac{1}{\delta t} \sum_{i=1}^n \frac{\delta t}{x_i^p} \approx \frac{1}{\delta t} \int_{t_0}^{t_n} \frac{dt}{(b-t)^p} \\ &= -\frac{(x_n - x_0)}{\delta t} \quad \text{for } p = 0 \\ &= \frac{1}{\delta t} \log \frac{x_0}{x_n} \quad \text{for } p = 1 \\ &= \frac{1}{\delta t} \left[\frac{1}{x_0^{p-1}} - \frac{1}{x_n^{p-1}} \right] \frac{1}{1-p} \quad \text{for } p = 2, 3 \text{ and } 4. \end{aligned}$$

Putting $x_0/x_n = \eta$, we get

$$\sum_{i=1}^n 1 = \frac{x_n}{\delta t} (\eta - 1)$$

$$\sum_{i=1}^n \frac{1}{x_i} = \frac{1}{\delta t} \log \eta$$

$$\sum_{i=1}^n \frac{1}{x_i^2} = \frac{1}{\delta t x_0} [\eta - 1]$$

$$\sum_{i=1}^n \frac{1}{x_i^3} = \frac{1}{2\delta t x_0^2} [\eta^2 - 1]$$

$$\sum_{i=1}^n \frac{1}{x_i^4} = \frac{1}{3\delta t x_0^3} [\eta^3 - 1] .$$

Substituting for the above in the determinant D we get

$$D = \begin{vmatrix} \frac{x_n(\eta - 1)}{\delta t} & \frac{o_0(\eta - 1)}{\delta t x_0} & \frac{1}{\delta t} \log \eta \\ \frac{o_0(\eta - 1)}{\delta t x_0} & \frac{o_0^2(\eta^3 - 1)}{\delta t 3x_0^3} & \frac{o_0(\eta^2 - 1)}{\delta t 2x_0^2} \\ \frac{1}{\delta t} \log \eta & \frac{o_0(\eta^2 - 1)}{\delta t 2x_0^2} & \frac{(\eta - 1)}{\delta t x_0} \end{vmatrix}$$

Expanding this determinant in terms of the elements of the first row and their corresponding co-factors we get

$$\begin{aligned} D = & \frac{o_0^2}{\delta t^3} \left\{ x_n(\eta - 1) \left[\frac{(\eta - 1)(\eta^3 - 1)}{3x_0^4} - \frac{(\eta^2 - 1)^2}{4x_0^4} \right] \right. \\ & - \frac{(\eta - 1)}{x_0} \left[\frac{(\eta - 1)^2}{x_0^2} - \frac{(\eta^2 - 1)}{2x_0^2} \log \eta \right] + \\ & \left. + \log \eta \left[\frac{(\eta - 1)(\eta^2 - 1)}{2x_0^3} - \frac{(\eta^3 - 1)}{3x_0^3} \log \eta \right] \right\} . \quad (20) \end{aligned}$$

Similarly the co-factors of the diagonal elements are

$$\begin{aligned} A_{11} &= \frac{o_o^2}{\delta t^2 x_o^4} \left\{ \frac{(\eta - 1)(\eta^3 - 1)}{3} - \frac{(\eta^2 - 1)^2}{4} \right\} \\ &= \frac{o_o^2}{\delta t^2 x_o^4} \frac{(\eta - 1)^4}{12} \end{aligned} \quad (21)$$

$$A_{22} = \frac{1}{\delta t^2} \left\{ \frac{(\eta - 1)^2}{\eta} - [\log \eta]^2 \right\} \quad (22)$$

$$A_{33} = \frac{o_o^2}{\delta t^2 x_o^2} \left\{ \frac{(\eta - 1)(\eta^3 - 1)}{3\eta} - (\eta - 1)^2 \right\} . \quad (23)$$

Now

$$\frac{1}{\eta} = \frac{x_n}{x_o} - 1 + 1 = \frac{x_n - x_o}{x_o} + 1$$

therefore

$$\frac{1 - \eta}{\eta} = \frac{b - t_n - b + t_o}{x_o}$$

therefore

$$x_o = (t_n - t_o) \frac{\eta}{\eta - 1} \quad (24)$$

Substituting for x_o in equations (21) to (23) and then substituting for A_{11} , A_{22} and A_{33} in equations (15) to (17) we get

$$\frac{s_a}{s} \sqrt{\frac{(t_n - t_o)}{\delta t}} = \sqrt{\frac{(\eta - 1)^5}{12\eta}} \quad , \quad (25)$$

$$\frac{s_b}{s} \sqrt{\frac{o_o^2}{\delta t(t_n - t_o)^3}} = \sqrt{\frac{\frac{\eta^2}{(\eta - 1)} - \frac{\eta^3}{(\eta - 1)^3} [\log \eta]^2}{c}} \quad , \quad (26)$$

and

$$\frac{s_o}{s} \sqrt{\frac{1}{\delta t(t_n - t_o)}} = \sqrt{\frac{\frac{1}{3}(\eta^3 - 1) - \eta(\eta - 1)}{c}} \quad , \quad (27)$$

$$\text{where } c^2 = \left\{ \frac{(\eta - 1)^5}{12\eta} - (\eta - 1)^3 + (\eta - 1)(\eta^2 - 1) \log \eta - \frac{(\eta^3 - 1)}{3} [\log \eta]^2 \right\} .$$

It will be seen that the right hand sides of the above three equations are functions of η only. The left-hand sides contain the standard deviations of the coefficients s_a , s_b and s_o together with quantities which are known or which can be arbitrarily chosen. Thus the parameters on the left-hand side may be conveniently plotted as functions of η as in Figs.1 to 3.

In predicting the standard deviation of the coefficients, η may be interpreted as the local mass ratio defined by

$$\eta = \frac{\text{mass of missile at start of the analysis}}{\text{mass of missile at end of the analysis}}$$

therefore

$$\eta = \frac{m_o - \dot{m}t_o}{m_o - \dot{m}t_n} = \frac{b - t_o}{b - t_n} = \frac{x_o}{x_n} \quad (28)$$

It is assumed here that the nominal values of m_o , \dot{m} and c_o will be used when predicting the accuracy which may be obtained so that the assumption of constant mass flow throughout the whole of the boosted flight will legitimately apply.

The quantity s^2 will depend on errors from all sources, e.g. reading error, radio propagation errors, transducer errors etc. However, prior experience with similar telemetry and transducer installations will afford information on the likely value of s which may be realised.

7 CASE WHERE 'a' IS KNOWN

In this case and in the notation of the previous section, we may write

$$y'_i = y_i - a$$

and

$$Y'_i = Y_i - a \quad .$$

Because 'a' is known, y'_i will be known, so that equation (4) becomes

$$v_i = y'_i - \frac{o}{b - t_i} \quad (29)$$

Putting $b = b_o - \beta$ and $c = c_o + \gamma$, the above equation reduces to

$$v_i = y'_i - \frac{c_o}{x_i} - \frac{1}{x_i} \gamma - \frac{o_o}{x_i^2} \beta \quad (30)$$

Squaring this equation and summing for all available values of i , and then partially differentiating with respect to β and γ in turn we get

$$o_o^2 \beta \sum_1^n \frac{1}{x_i^4} + o_o \gamma \sum_1^n \frac{1}{x_i^3} = \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{o_o}{x_i^2} \quad (31)$$

$$o_o \beta \sum_1^n \frac{1}{x_i^3} + \gamma \sum_1^n \frac{1}{x_i^2} = \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{1}{x_i} \quad (32)$$

where, as before, $x_i = b_o - t_i$.

The solution of these equations is

$$D'\beta = \left\{ \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{o_o}{x_i^2} \right\} \left\{ \sum_1^n \frac{1}{x_i^2} \right\} - \left\{ \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{1}{x_i} \right\} \left\{ \sum_1^n \frac{o_o}{x_i^3} \right\} \quad (33)$$

$$D'\gamma = \left\{ \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{1}{x_i} \right\} \left\{ \sum_1^n \frac{o_o^2}{x_i^4} \right\} - \left\{ \sum_1^n \left(y_i' - \frac{o_o}{x_i} \right) \frac{o_o}{x_i^2} \right\} \left\{ \sum_1^n \frac{o_o}{x_i^3} \right\} \quad (34)$$

where

$$D' = \begin{vmatrix} o_o^2 \sum_1^n \frac{1}{x_i^4}, & o_o \sum_1^n \frac{1}{x_i^3} \\ o_o \sum_1^n \frac{1}{x_i^3}, & \sum_1^n \frac{1}{x_i^2} \end{vmatrix} \quad (35)$$

The method of repeated adjustment of b_o and o_o is the same as in the general case.

The estimate of the variances of the coefficients are given by

$$s_b^2 = \frac{s^2 \sum_1^n \frac{1}{x_i^2}}{D'} \quad (36)$$

$$s_o^2 = \frac{s^2 \sum_1^n \frac{o_o^2}{x_i^4}}{D'} \quad (37)$$

where

$$s^2 = \frac{E}{(m-3)D} = \frac{\sum_{i=1}^n v_i^2}{m-3} \quad (38)$$

and

$$E = \begin{vmatrix} o_o^2 \sum_{i=1}^n \frac{1}{x_i^4} & , & o_o \sum_{i=1}^n \frac{1}{x_i^3} & , & \sum_{i=1}^n \left(y_i - \frac{o_o}{x_i}\right) \frac{o_o}{x_i^2} \\ o_o \sum_{i=1}^n \frac{1}{x_i^3} & , & \sum_{i=1}^n \frac{1}{x_i^2} & , & \sum_{i=1}^n \left(y_i - \frac{o_o}{x_i}\right) \frac{1}{x_i} \\ o_o \sum_{i=1}^n \left(y_i - \frac{o_o}{x_i}\right) \frac{1}{x_i^2} & , & \sum_{i=1}^n \left(y_i - \frac{o_o}{x_i}\right) \frac{1}{x_i} & , & \sum_{i=1}^n \left(y_i - \frac{o_o}{x_i}\right)^2 \end{vmatrix} \quad (39)$$

Making use of the approximations derived in section 6 for the summations, the expressions which may be used to predict the accuracy of the coefficients are in this case

$$\frac{s_b}{s} \sqrt{\frac{o_o^2}{\delta t(t_n - t_o)^3}} = \sqrt{\frac{12\eta^3}{(\eta - 1)^6}} \quad (40)$$

$$\frac{s_o}{s} \sqrt{\frac{1}{\delta t(t_n - t_o)}} = \sqrt{\frac{4\eta(\eta^3 - 1)}{(\eta - 1)^5}} \quad (41)$$

The left-hand sides are the same as the parameters on the left-hand sides of the corresponding equations (26) and (27) in the general case. Similarly the right-hand sides are functions of η . Equations (40) and (41) are represented graphically in Figs. 4 and 5.

The variances of b and c given by equations (40) and (41) apply only when 'a' is known precisely. In practice, the pre-flight measurement of 'a' will be subject to errors, while further bias error effectively altering 'a' will occur in flight. Then error in the knowledge of 'a' will propagate errors in both b and c . Let the error be δa . Therefore $y_i' = y_i - a - \delta a$. Thus the error propagated in b and c may be found by substituting this expression for y_i' in equations (33) and (34) and separating out that part containing δa . Thus

$$\delta\beta = \frac{\delta a \left\{ \left[\sum_{i=1}^n \frac{o_o^2}{x_i^2} \right] \left[\sum_{i=1}^n \frac{1}{x_i^2} \right] - \left[\sum_{i=1}^n \frac{1}{x_i} \right] \left[\sum_{i=1}^n \frac{o_o}{x_i^3} \right] \right\}}{D'} \quad (42)$$

$$\delta\gamma = \frac{\delta a \left\{ \left[\sum_{i=1}^n \frac{1}{x_i} \right] \left[\sum_{i=1}^n \frac{o_o^2}{x_i^4} \right] - \left[\sum_{i=1}^n \frac{o_o}{x_i^2} \right] \left[\sum_{i=1}^n \frac{o_o}{x_i^3} \right] \right\}}{D'} \quad (43)$$

Making use of the approximations derived in section 6 these equations reduce to

$$\frac{\delta\beta}{\delta a} \frac{o_o}{(t_n - t_o)^2} = \frac{12\eta^2}{(\eta-1)^6} \left\{ (\eta-1)^2 - \frac{1}{2}(\eta^2-1) \log \eta \right\} \quad (44)$$

$$\frac{\delta\gamma}{\delta a(t_n - t_o)} = \frac{\eta}{(\eta-1)^5} \left\{ 4(\eta^3-1) \log \eta - 6(\eta-1)(\eta^2-1) \right\} \quad (45)$$

These equations are represented graphically in Figs.6 and 7.

8 DISCUSSION

In practice, the usefulness of the method of determining the missile parameters, b and c , will depend mainly on the accuracy with which they can be found. This section, therefore, will be concerned chiefly with an appreciation of those factors which have a significant effect on the accuracy.

The curves given in Figs.1 to 7 may be used to assess, approximately, the accuracy which can be obtained by applying the method of analysis to a missile of known nominal performance. Although these curves are of general application, they do not show clearly how the accuracy is affected by the time of the start of the analysis, t_o , and the period of the analysis

$t_n - t_o$. This arises from the fact that the local mass ratio, η , is dependent upon both t_o and $t_n - t_o$. Consequently, the sensitivity to $t_n - t_o$ is hidden, to a certain extent, by the choice of η as the independent variable in Figs.1 to 7. In order to illustrate this sensitivity, Figs.8 and 9 have been prepared as a particular example. These figures show the variation of the accuracy of determination of b and c with t_o . In this example it was assumed that:-

- (i) δt was 0.01 sec,
- (ii) s was 1% of full scale,

- (iii) the nominal value of b was 135 seos
 and (iv) the nominal value of c was 15 seo.

All of these values were considered to be reasonably typical of a medium range ballistic missile such as Blue Streak. Three separate cases were considered, as follows:-

- (a) 'a' unknown,
 (b) 'a' known to a standard deviation of 1% of full scale,
 and (c) 'a' known precisely.

For (a) and (b) above, curves for $t_n - t_o$ equal to 10 seos and 20 seos are given. For (c), a curve for $t_n - t_o$ equal to 20 seos only is given.

It will be seen from the curves given in these figures that the standard deviations of b and c decrease rapidly as t_o is increased. Similarly, for constant t_o , the standard deviations decrease even more rapidly as $t_n - t_o$ is increased. These curves show that accurate values of b and c will only be obtained with large values of t_o and $t_n - t_o$, that is towards the end of the boost phase. The analysis is also restricted to the later stage of the boost phase by the assumption of zero aerodynamic forces. This restriction means that the analysis will only be applied to times when the acceleration is changing rapidly and, as a consequence, when the mass flow and thrust are most likely to be changing. The effect of the changing pump inlet pressures (arising from the changing missile acceleration) on the mass flow and thrust can be predicted using the known, nominal engine characteristics. The period of the analysis must then be chosen so that the mass flow and thrust are constant to within approximately 0.25%. This permissible period of analysis will be large when the missile has just left the atmosphere and will decrease towards cut-off. In practice, therefore, the accuracy is likely to be reasonably constant over the later stage of boosted flight, the gain in accuracy due to increasing t_o being balanced by the decrease in accuracy due to the reduction in $t_n - t_o$.

If 'a' is known precisely, a considerable improvement in accuracy may be realised compared with that obtaining when 'a' is unknown. For instance, in the particular example given in Figs.8 and 9, when $t_o = 150$ seos, and $t_n - t_o = 20$ seos a precise knowledge of 'a' improves the accuracy by a factor of 8 for b and 16 for c compared with the case when 'a' is unknown. In practice, 'a' can never be known precisely even with very accurate pre-flight calibration because any bias errors arising in the telemetry system during flight must inevitably be included in the coefficient 'a'. If 'a' is known to a standard deviation of 1% of full scale, a general improvement in accuracy will result compared with the case of 'a' unknown as is shown by the curves in Figs.8 and 9. The main effect appears to be that the sensitivity of the accuracy to t_o is reduced. It should be noted, however, that for the larger values of t_o , the two curves cross over so that an analysis assuming that 'a' is unknown leads to better accuracy than that when 'a' is known only to 1% standard deviation. This is, of course, due to the fact that in these circumstances the method of analysis determines 'a' to a standard deviation of better than 1%.

If p is defined as the inverse of the time interval between successive readings, δt , then p will be equal to the numbers of readings per sec, or, as it is usually termed, the sampling frequency. Replacing δt by p in equations (25), (26) and (27) shows that the standard deviations, s_a , s_b and s_c are proportional to the inverse of the square root of the sampling frequency. Figs. 8 and 9 were prepared assuming a sampling frequency of 100 readings per sec which roughly corresponds to the switch speed of current telemetry systems. The maximum accuracy will be obtained if all of the available readings are used. However, as the use of high speed computers is envisaged, the numbers of readings employed in the calculation is relatively unimportant.

The coefficient c appears only in the expression for s_b , the standard deviation of b . To a limited extent, $c (= Ig/k)$ may be adjusted by the choice of k , the accelerometer scale factor. As s_b is inversely proportional to c , it follows that k should be chosen as small as possible in order that s_b should be as small as possible. Normally only one accelerometer is used, its range being chosen to cover the whole range of accelerations which are likely to occur in the boost phase. This imposes a lower limit on the choice of k . The use of two or more accelerometers to cover consecutive phases of the boost phase enables a smaller value of k to be chosen but this will only have advantage if the errors introduced by missile vibration are not increased (increasing s) as k is reduced. The accelerometer scale factor, k , will be measured by the pre-flight calibration of the system. An estimate of the standard deviation of k must also be made. The estimated standard deviation of the determination of the specific impulse will therefore be the vectorial sum of the standard deviations of c and k .

9 CONCLUSIONS

The method as presented provides an accurate method of determining the vacuum specific impulse of a missile from longitudinal acceleration records provided that the engine performance is reasonably constant. In order to achieve as high an accuracy as possible, δt should be chosen as small as possible, that is the sampling frequency should be as high as possible, $(t_n - t_0)$ the period of flight to be analysed should be chosen as large as possible consistent with the assumption of constant mass flow and k the scale of the accelerometer should be chosen as small as possible. The accuracy of the method in the case where 'a', the zero bias of the accelerometer is known is very much better than that obtaining in the case where 'a' is unknown. 'k' should be measured as accurately as possible by pre-flight ground testing. The method is useful in that it provides an overall check on the performance of the rocket engine and unlike other methods does not assume knowledge of the engine operating characteristics other than that the engine is running steadily.

The method can be used directly to derive an accurate measurement of the parameter m/\dot{m} . Combining this with direct measurement of mass flow, e.g. by flow-meter, can give a better measurement of missile mass in the later stages of boost flight than could be obtained by integration of mass flow from launch.

Fundamentally, there is no reason why the mass flow or the specific impulse need be assumed constant except to simplify the analysis. To extend the method to cover the case when the engine performance is varying, it is only necessary to assume that the mass flow and the specific impulse can be expressed as polynomials in terms of t , the time variable. The method of

least squares fitting by the process of differential adjustment, although more complicated, will then evaluate the coefficients of the polynomials. Extension of the method on these lines would enable the method to apply to missiles such as Blue Streak which employ a system of thrust reduction during the later stages of the boost flight.

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<u>No.</u>	<u>Author</u>	<u>Title, etc.</u>
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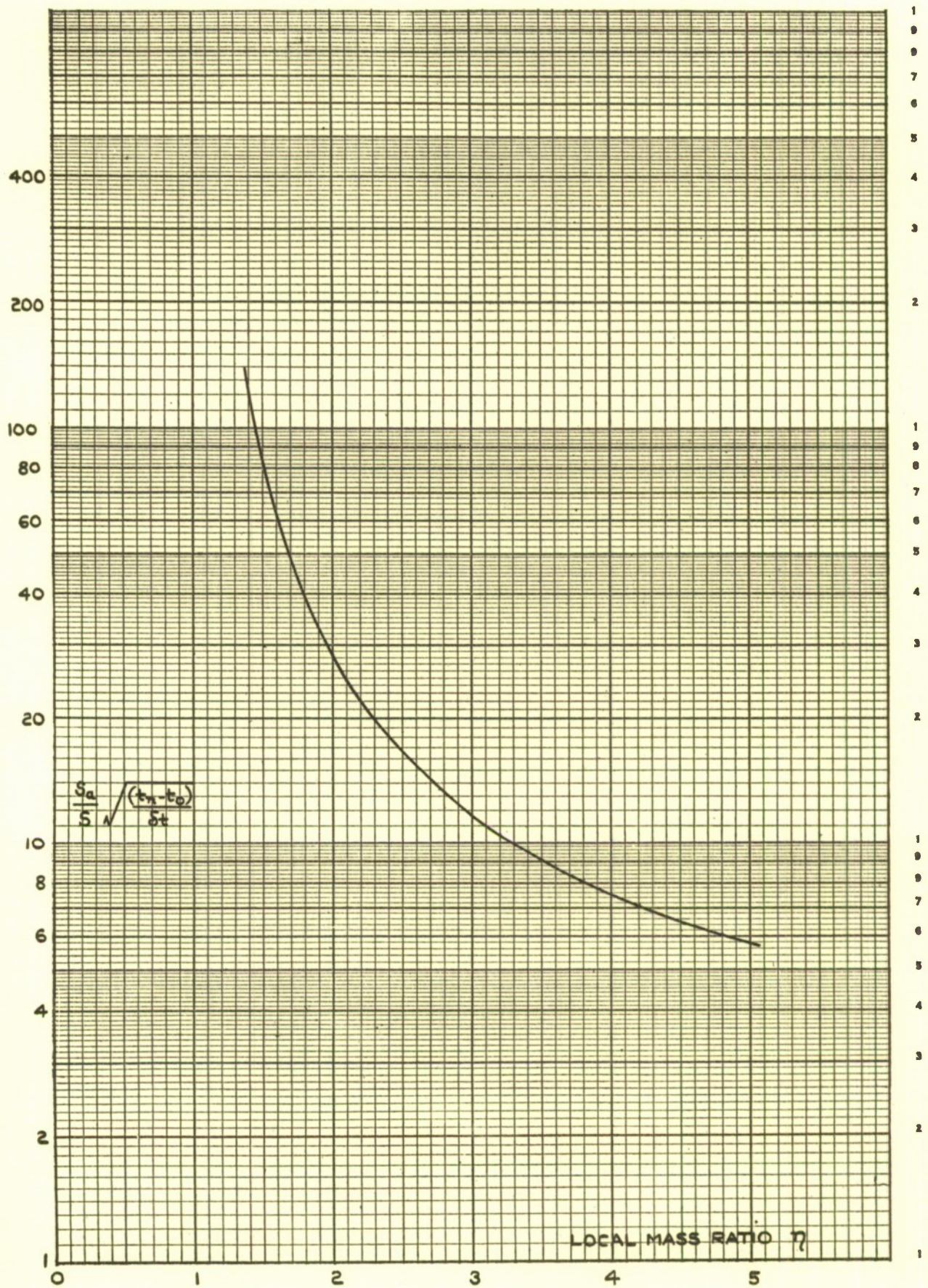


FIG. I. VARIATION OF $\frac{S_a}{S} \sqrt{\frac{(t_n - t_o)}{\delta t}}$ WITH THE LOCAL MASS RATIO.

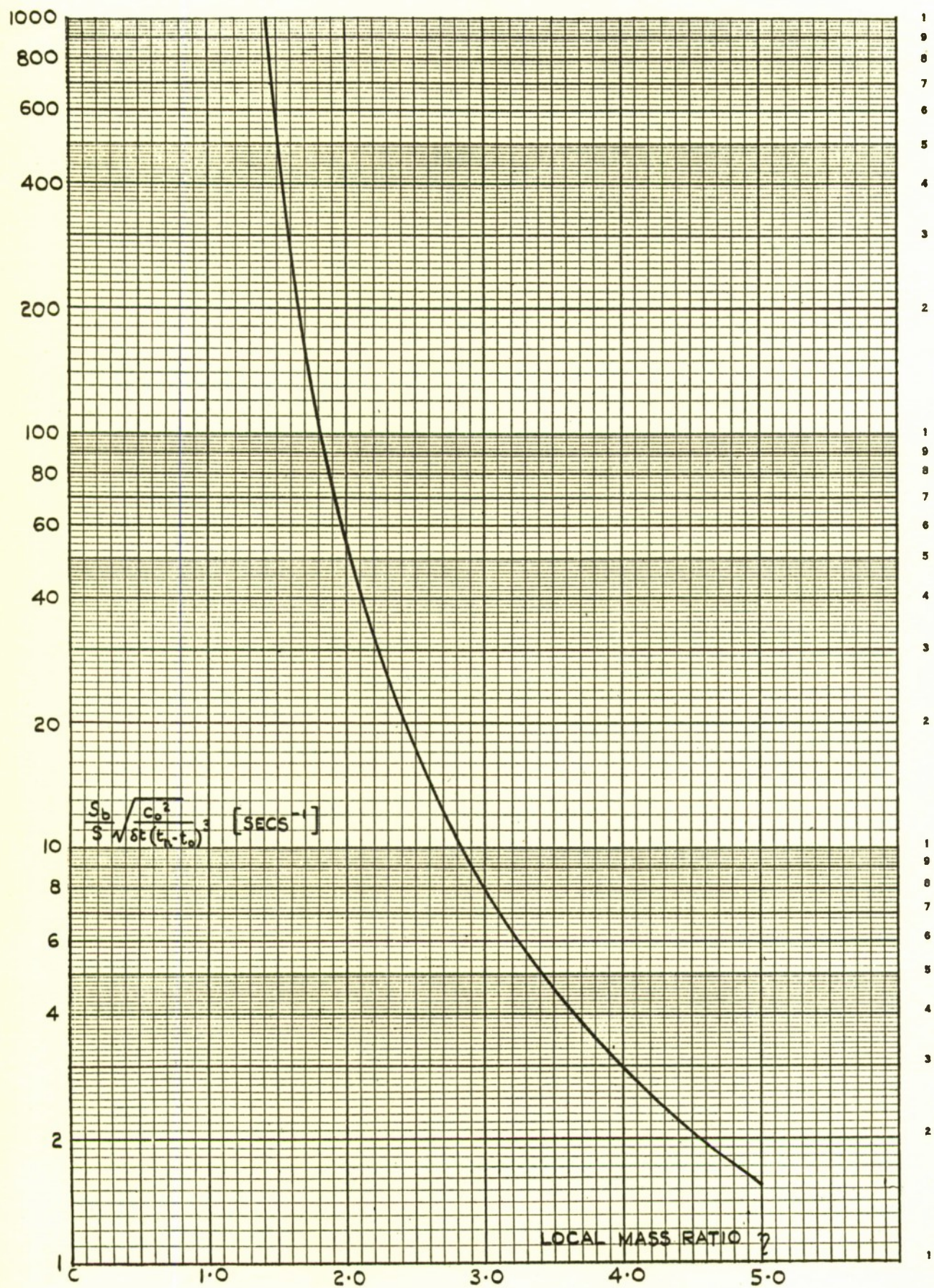


FIG. 2. VARIATION OF $\frac{S_b}{S} \sqrt{\frac{C_o^2}{8t(t_n - t_o)^3}}$ WITH THE LOCAL MASS RATIO, 'a' UNKNOWN.

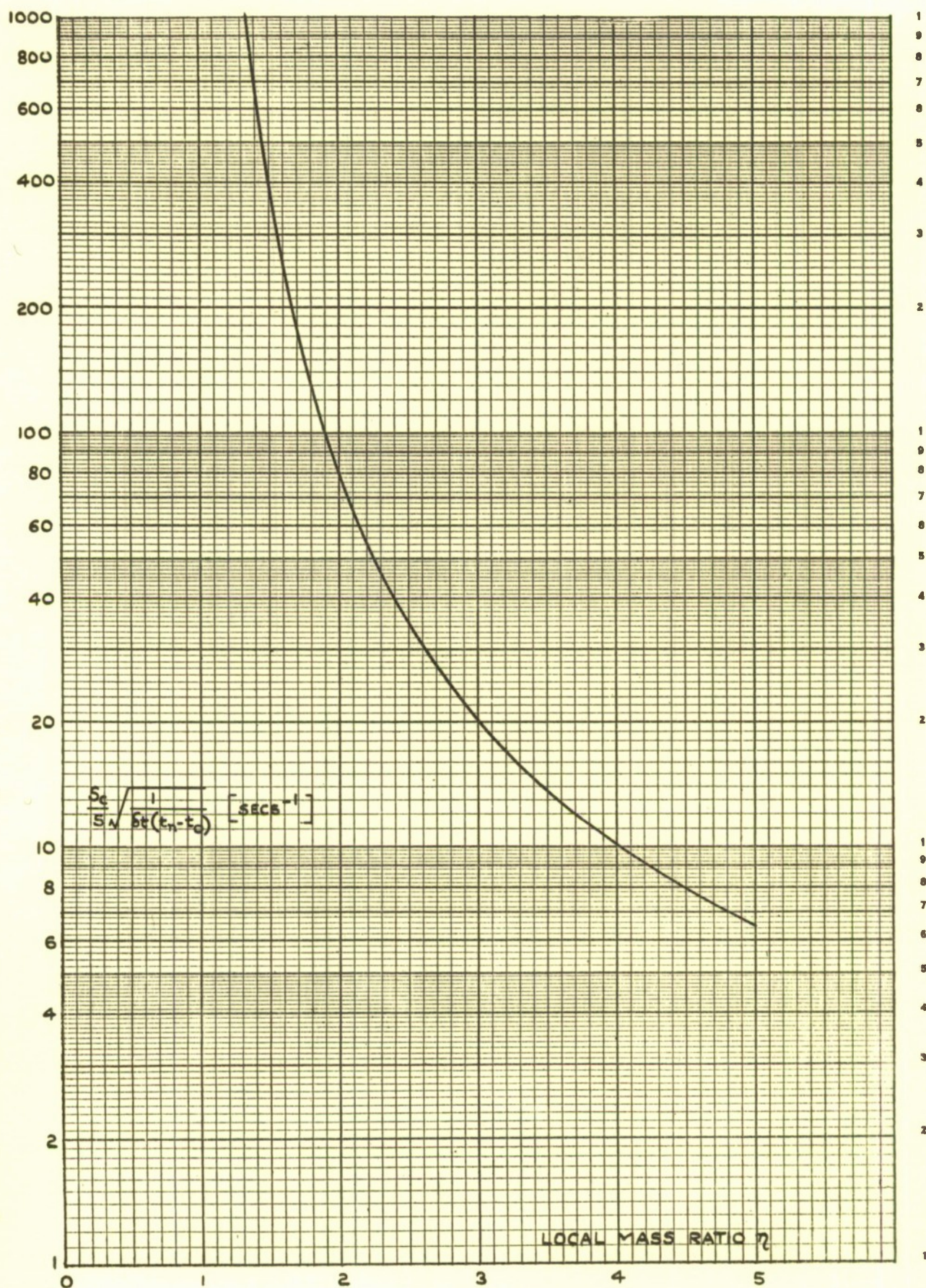


FIG. 3. VARIATION OF $\frac{S_c}{S} \sqrt{\frac{1}{\delta t (t_n - t_0)}}$ WITH THE LOCAL MASS RATIO, 'a' UNKNOWN.

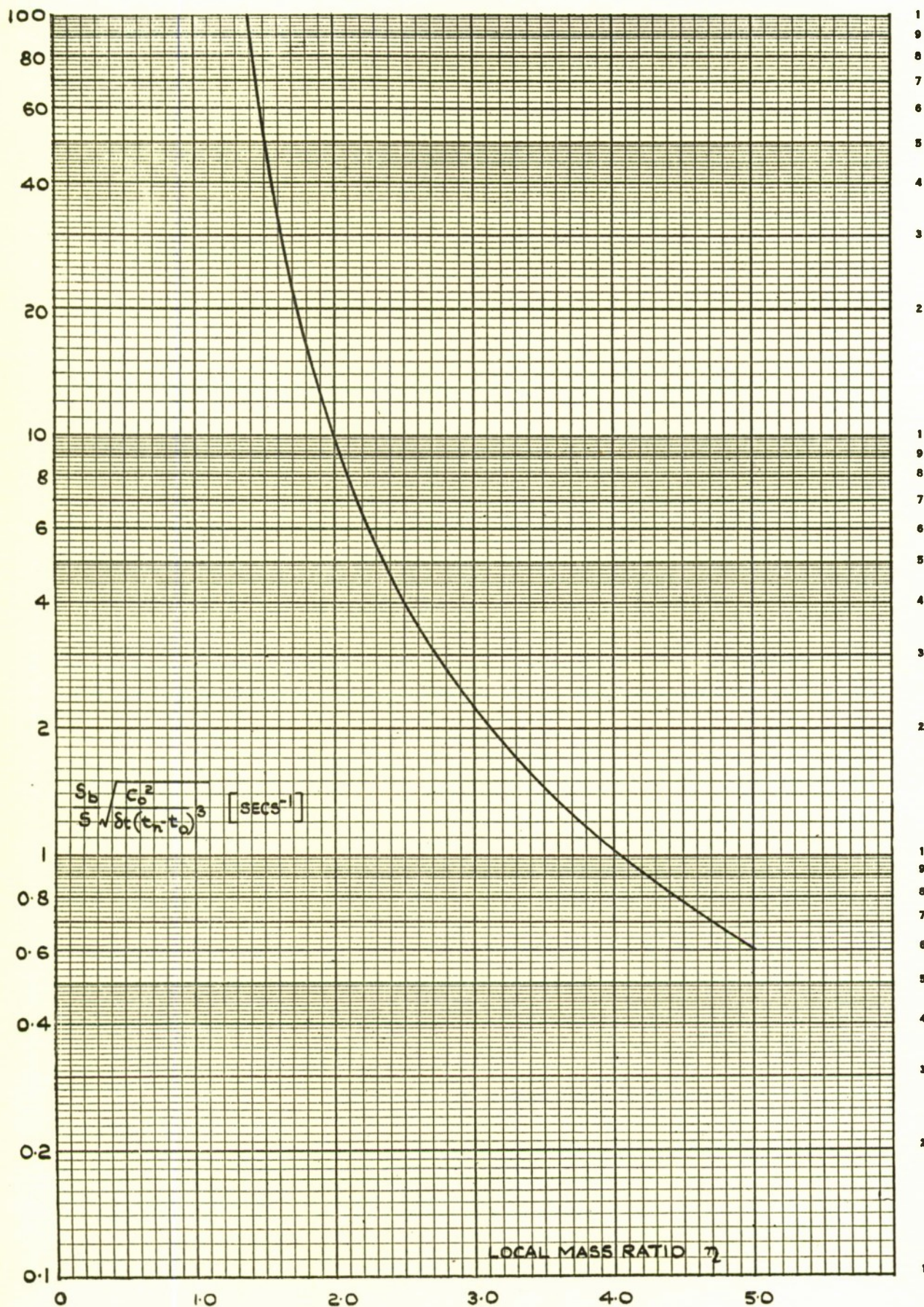


FIG. 4. VARIATION OF $\frac{S_b}{S} \sqrt{\frac{C_o^2}{\delta t (t_n - t_o)^3}}$ WITH THE LOCAL MASS RATIO, 'a' KNOWN.

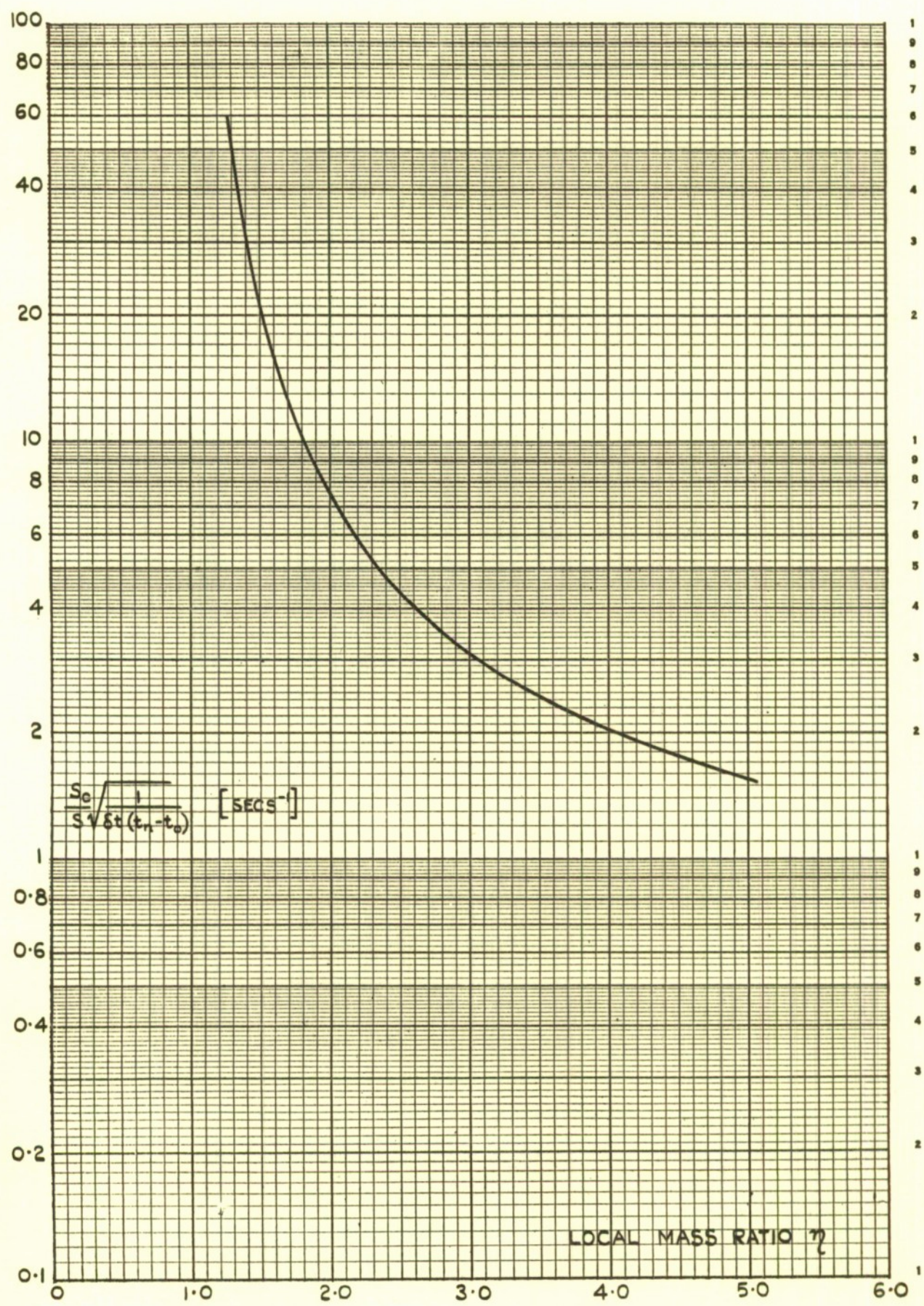


FIG.5. VARIATION OF $\frac{S_c}{S \sqrt{\delta t (t_n - t_0)}}$ WITH THE LOCAL MASS RATIO, 'a' KNOWN.

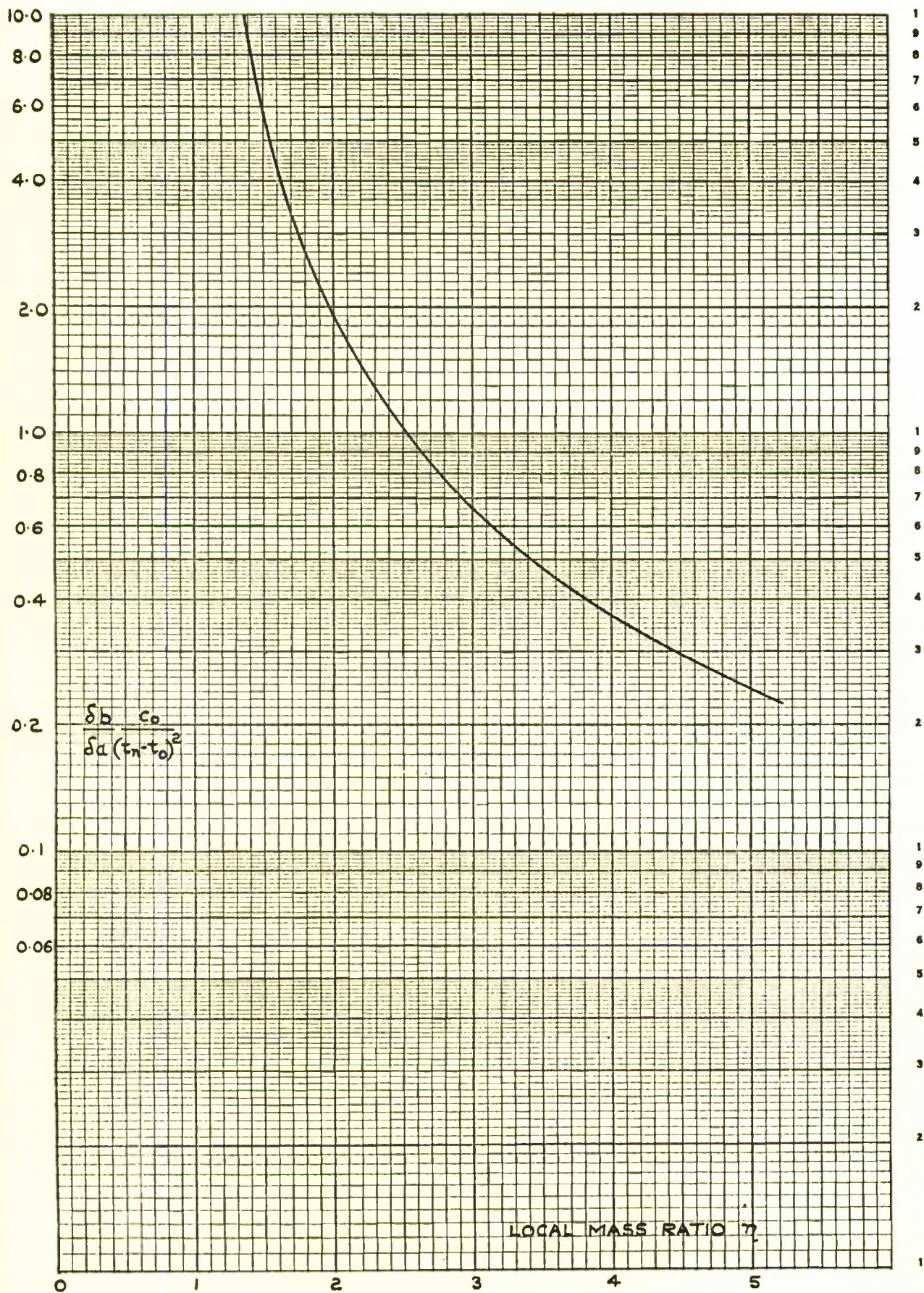


FIG. 6. VARIATION OF $\frac{\delta b}{\delta a} \frac{c_0}{(t_n - t_0)^2}$ WITH THE LOCAL MASS RATIO, 'a' SUBJECT TO ERROR δa .

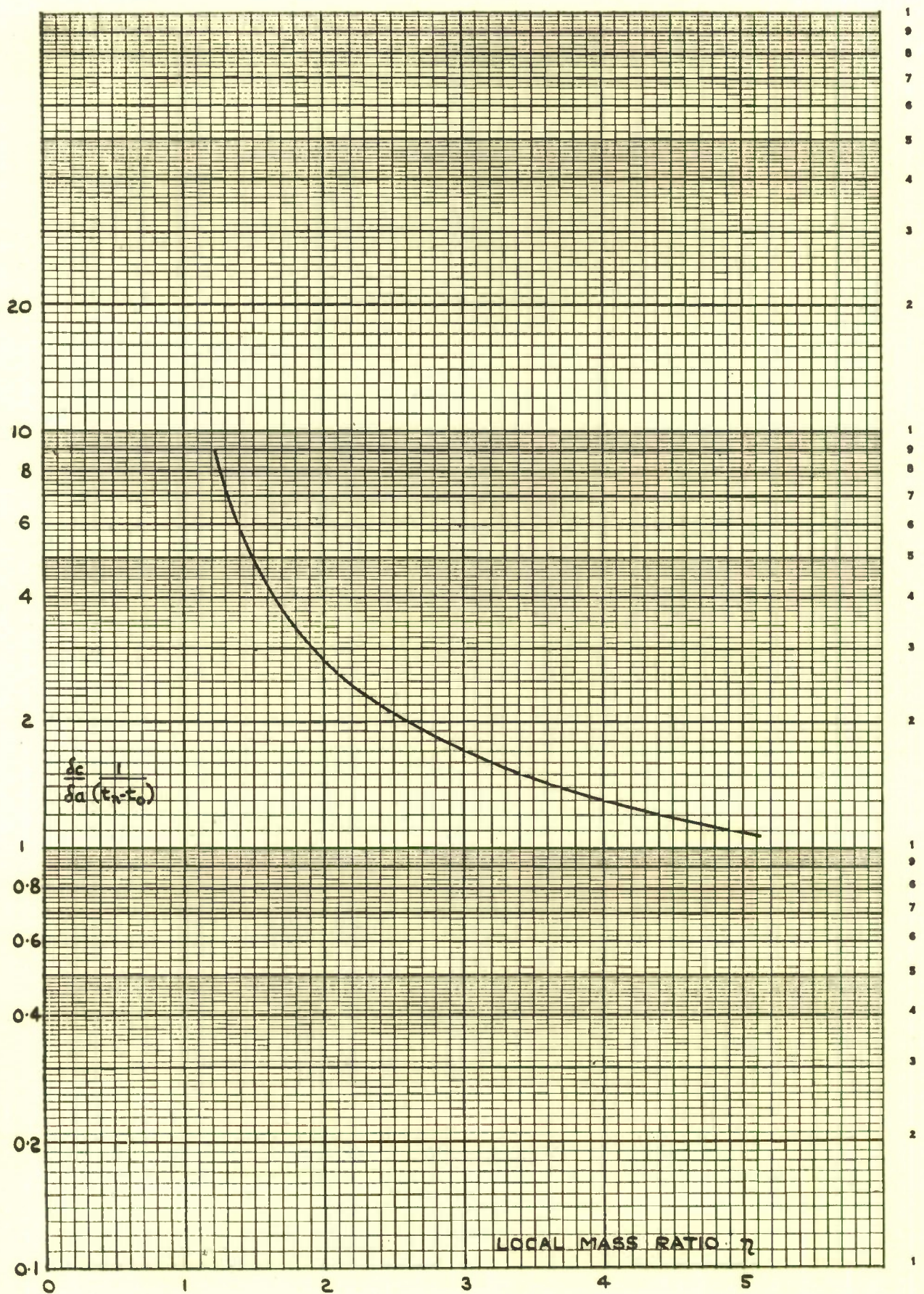


FIG. 7. VARIATION OF $\frac{\delta c}{\delta a} \frac{1}{(t_n - t_0)}$ WITH THE LOCAL MASS RATIO, 'a' SUBJECT TO ERROR δa .

FIG. 8.

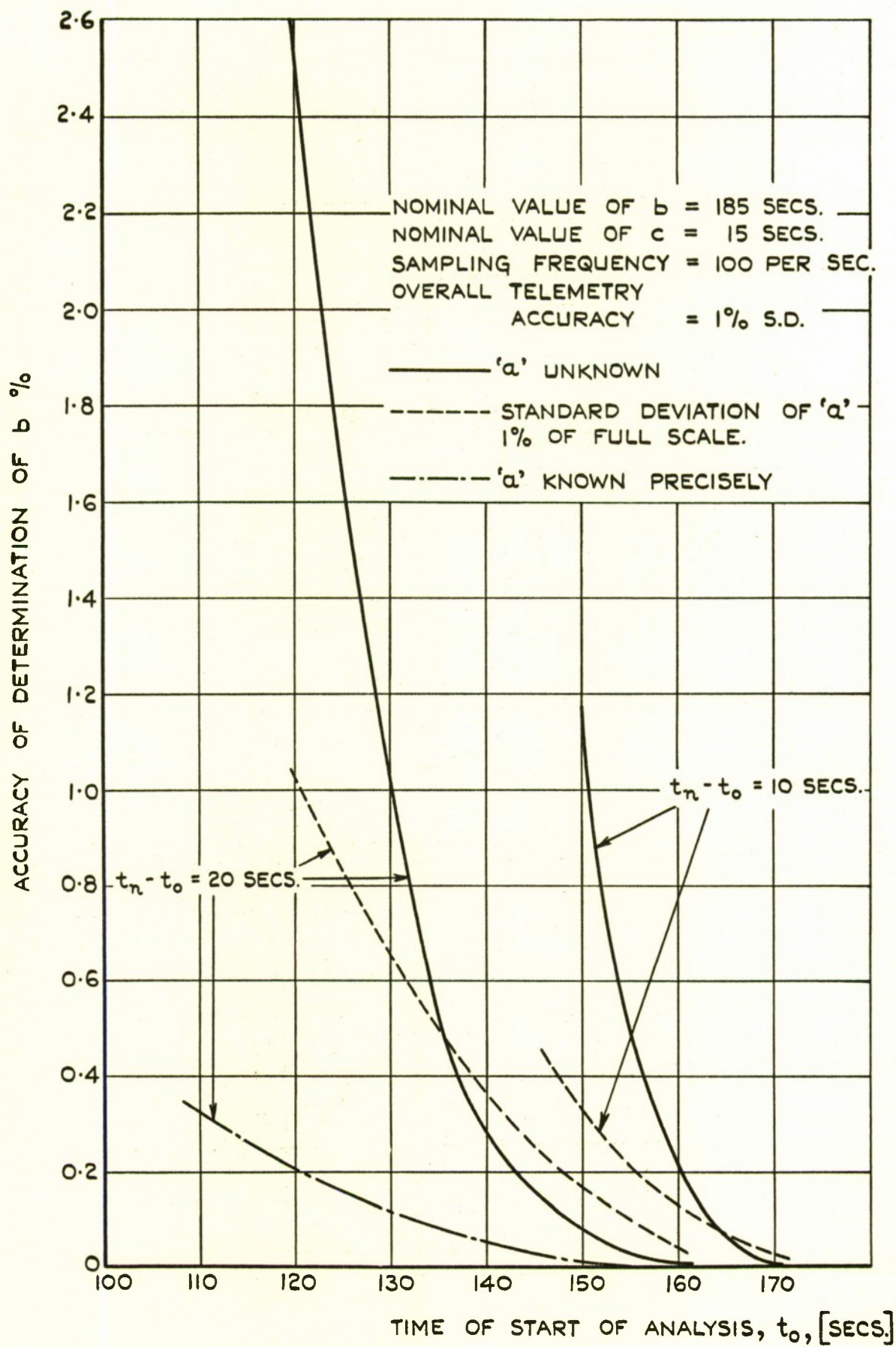


FIG. 8. VARIATION OF THE ACCURACY OF DETERMINATION OF 'b' WITH TIME OF START OF ANALYSIS.

FIG. 9.

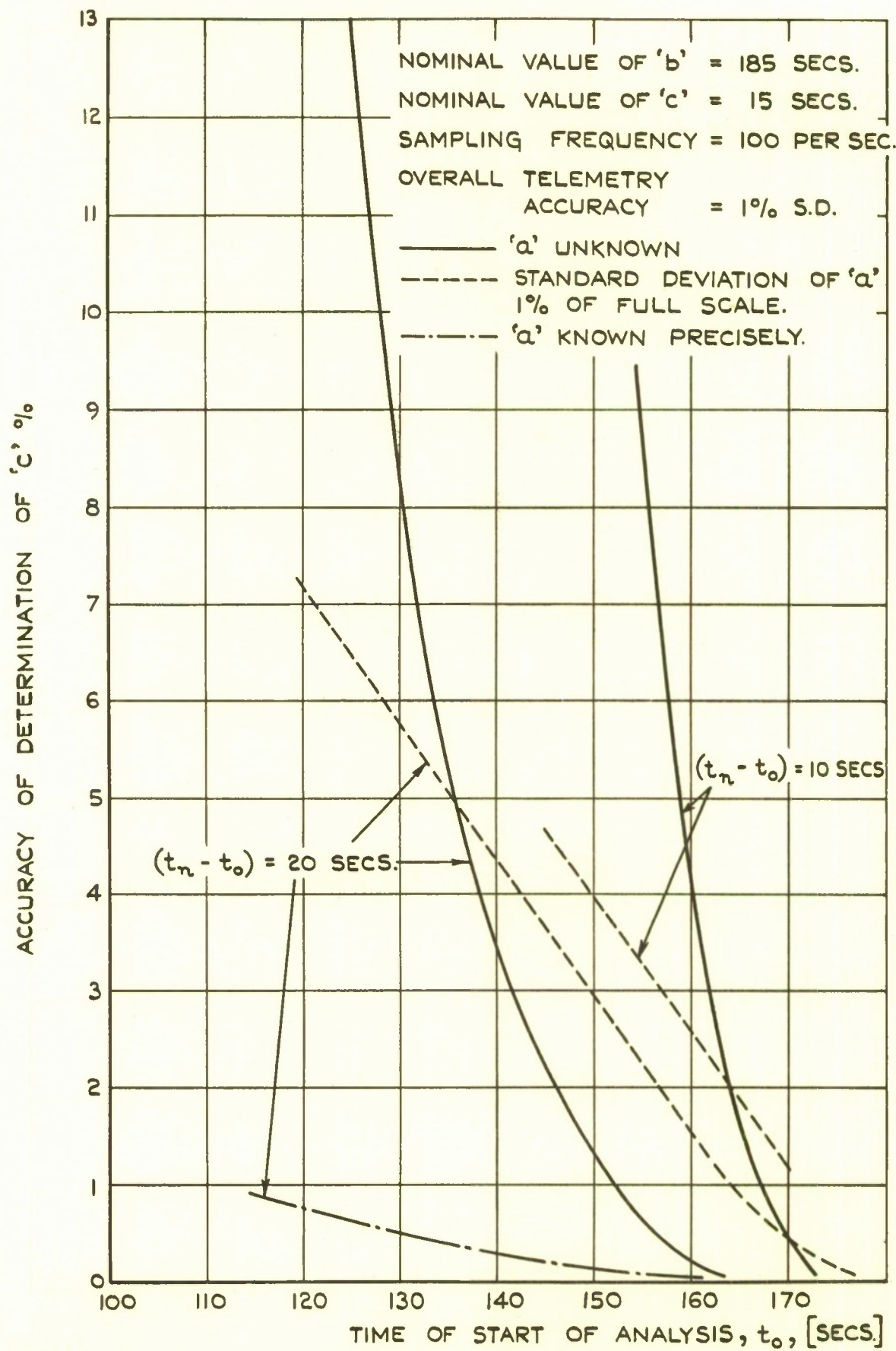


FIG. 9. VARIATION OF THE ACCURACY OF DETERMINATION OF 'c' WITH TIME OF START OF ANALYSIS.

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